ISRAEL JOURNAL OF MATHEMATICS, Vol. 30, Nos. 1-2, 1978

WEAK CONVERGENCE OF TRAJECTORIES OF NONEXPANSIVE SEMIGROUPS IN HILBERT SPACE

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ABSTRACT

Let $S: I \times X \to X$ be a nonexpansive semigroup on a weakly closed (not necessarily convex) subset X of a real Hilbert space E. In this note we present a theorem on the weak convergence of a trajectory $\{S(t,x)\}_{t \in I}$ together with a very simple and elementary proof, which extends and unifies several recent results due to Baillon, Bruck, Pazy and Reich.

Let E be a real Hilbert space with inner product $\langle x, y \rangle$ and norm $\|x\| = \langle x, x \rangle^{1/2}$, X be a weakly closed (not necessarily convex) subset of E and I an unbounded subset of $[0, \infty)$ such that

$$(1) \qquad t + h \in I \quad \text{for all } t, h \in I$$

and

$$(2) \qquad t - h \in I \quad \text{for all } t, h \in I \text{ with } t > h$$

(e.g. $I = [0, \infty)$ or $I = \mathbf{N}$).

Let furthermore S be a mapping of $I \times X$ into X such that

$$(3) \qquad S(t+h, x) = S(t, S(h, x)) \quad \text{for all } t, h \in I \text{ and } x \in X$$

and

$$(4) \qquad \|S(t,x) - S(t,y)\| \leqq \|x - y\| \quad \text{for all } t \in I \text{ and } x, y \in X$$

(i.e., S is a (not necessarily continuous) nonexpansive semigroup).

The fixed point set F of S is defined by

$$F := \{y \in X : S(t,y) = y \text{ for all } t \in I\}.$$

Received September 1, 1977

THEOREM. *If F is nonempty and $x \in X$, then the following conditions are equivalent*:

(a) $\{S(t,x)\}_{t\in I}$ *converges weakly to a fixed point of* S,

(b) $\{S(t,x)\}_{t\in I}$ *converges weakly*,

(c) $\{S(t,x)-S(t+h,x)\}_{t\in I}$ *converges weakly to* 0 *for all* $h \in I$,

(d) *If* $\omega_w(x)$ *is defined by*

$$\omega_w(x):=\{y \in X\colon y = w-\lim_{n\to\infty}\{S(t_n,x)\}$$

$$\text{for some strictly increasing sequence } \{t_n\}_{n\in N} \text{ such that } t_n \to \infty\},$$

then $\omega_w(x) \subset F$.

PROOF. We remark first that on account of (1)–(4), for all $y \in F$ the net $\{\|S(t,x)-y\|^2\}_{t\in I}$ is monotone non-increasing and hence convergent to a real number $\rho(y)$.

"(a) $\Rightarrow$ (b)": obvious. "(b) $\Rightarrow$ (c)": obvious. "(c) $\Rightarrow$ (d)": (Basic for this part is a simple refinement of an inequality which has already been used by Pazy in the proof of [3, theor. 3].)

Let $y \in X$ such that $y = w-\lim_{n\to\infty}\{S(t_n,x)\}$ for some strictly increasing sequence $\{t_n\}_{n\in N}$ in I with $t_n \to \infty$. Since F is nonempty, we can choose $z \in F$. Then for all $n \in \mathbf{N}$ and $h \in I$

$$\begin{aligned}
0 &\leqq \|S(t_n,x)-y\|^2 - \|S(t_n+h,x)-S(h,y)\|^2\\
&= \|S(t_n,x)-z\|^2 + 2\langle S(t_n,x)-z, z-y\rangle + \|z-y\|^2\\
&\quad - \|S(t_n+h,x)-z\|^2 - 2\langle S(t_n+h,x)-z, z-S(h,y)\rangle - \|z-S(h,y)\|^2\\
&= \|S(t_n,x)-z\|^2 - \|S(t_n+h,x)-z\|^2 + 2\langle S(t_n,x)-z, S(h,y)-y\rangle\\
&\quad + 2\langle S(t_n,x)-S(t_n+h,x), z-S(h,y)\rangle + \|z-y\|^2 - \|z-S(h,y)\|^2.
\end{aligned}$$

This yields, letting n tend to infinity, for all $h \in I$

$$0 \leqq 2\langle y-z, S(h,y)-y\rangle + \|z-y\|^2 - \|z-S(h,y)\|^2 = -\|y-S(h,y)\|^2,$$

i.e., $S(h,y)=y$. Therefore $y \in F$. (d) $\Rightarrow$ (a): For all $y_1, y_2 \in X$ and $t \in I$ we have

$$\|S(t,x)-y_1\|^2 - \|S(t,x)-y_2\|^2 = 2\langle S(t,x)-y_2, y_2-y_1\rangle + \|y_2-y_1\|^2.$$

Therefore the inclusion $\omega_w(x) \subset F$ gives

$$\rho(y_1)-\rho(y_2) = \|y_1-y_2\|^2 \qquad \text{for all } y_1, y_2 \in \omega_w(x).$$

But this implies that $\omega_w(x)$ consists of at most one point, which yields the assertion, since $\{\|S(t,x)\|\}_{t\in I}$ is bounded.

REMARK. If S is a discrete semigroup (i.e., $I = \mathbf{N}$ and $S(n,x) = f^n(x)$, where $f: X \to X$ is a nonexpansive self-mapping of X) and X is convex, then this theorem is due to Pazy [3] and Bruck [2]. If S is a continuous semigroup of contractions (cf. Pazy [4]) and X is convex, this theorem is due to Pazy [4] and Baillon, Bruck and Reich [1].

REFERENCES

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